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# $H_\infty$ filtering for systems with time-varying delay satisfying a certain stochastic characteristic

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**Abstract:** This study proposes a class of $H_\infty$ filter design for linear time-delay system. The time delay considered here is assumed to be satisfying a certain stochastic characteristic. Corresponding to the probability of the delay taking value in different intervals, a stochastic variable satisfying Bernoulli random binary distribution is introduced and a new system model is established by employing the information of the probability distribution. Then new criteria is derived for the filtering-error systems, which can lead to much less conservative analysis results. It should be noted that the solvability of the obtained criteria depend not only on the size of the delay, but also the probability distribution of it. At last, numerical examples are given to demonstrate the effectiveness and the merit of the proposed method.

## 1 Introduction

Filtering problem has wide applications in signal-processing, communications and control application. The problem of filtering can be briefly described as the design of an estimator from the measured output to estimate the state of the given systems. $H_\infty$ filtering was first introduced in [1]. Ever since, much work has been done for the design of $H_\infty$ filter, for example [2–5] and the references therein. One of its main advantages is that it is insensitive to the exact knowledge of the statistics of the noise signals.

Recently, the problem of $H_\infty$ filtering of time-delay systems has also received great attention because of the fact that for many practical filtering applications, time delays cannot be neglected in the procedure of filter design and their existence usually results in a poor performance [6–8]. Some useful results on $H_\infty$ filtering for time-delay systems have been reported in the literature and there are two kinds of results, namely delay-independent filtering [9] and delay dependent [10–13]. The delay-dependent results are usually less conservative, especially when the time delay is small.

From the stochastic theory point of view, the variation of the delay may often stick to some probability distribution, although it is fast time varying and not derivable. As pointed out in [14], for a given wireless network, it can be measured that there exists a small number $\varepsilon$ such that $\text{Prob}\{\tau(k) > d\} < \varepsilon$, where $d$ is a constant. For this case, what we need to investigate is, for a given $\varepsilon$, to find the upper bound for $d$, or, for a given $d$, to find the upper bound for $\varepsilon$. Obviously, the variation probability of the time delay will affect the size of the allowable variation range of the delay. Furthermore, in many real systems, such as the networked control systems, affected by the external disturbances and some unpredictable elements, the practical time-varying delay may have some abrupt burst, the delay in this case may be very large with a small probability, which is usually outside the allowable variation range obtained by the traditional methods in [15–22]. Therefore the system performance should depend not only on the variable range of the delay, but also the probability distribution of the delay. However, the information of the probability of the delay have been omitted by most of the researchers. Therefore a challenging issue is on how to derive some filter design criteria that can exploit the known probability distribution of the delay and obtain a larger allowable variation range of delay. To the best of the authors' knowledge, up to now, no results have been reported for the filter design when both the information of variation range of the time delay and the information of probability of the time-varying delay in an interval are taken into consideration.

Motivated by the above discussions, the aim of this paper is to consider the delay-distribution-dependent stability of the filtering-error system. The time-varying delay varies randomly in an interval and there are no constraints on the derivative of the delay. Based on the delay taking values in different intervals, a stochastic variable satisfying Bernoulli binary distribution is introduced to rebuild the following traditional system

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{A}_{\mathrm{d}}x(t - \tau(t)) + \boldsymbol{B}w(t) \tag{1}$$

Then a new type of system model with stochastic parameter matrix is proposed, the system (1) can be seen as a simplification of the system without considering the information of probability distribution and the abrupt burst of the delay. By using the Lyapunov–Krasovskii functional approach and the convexity of the matrix equations, delay-distribution-dependent criteria for the exponential mean-square stability of the filtering-error system are derived that are shown as a set of matrix inequalities. Examples used in [23, 24] are employed to show the effectiveness and less conservativeness of the proposed methods.

*Notation:* The notation used here is fairly standard. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of real $\boldsymbol{n}\times\boldsymbol{m}$ matrices, $\boldsymbol{I}$ is the identity matrix of appropriate dimensions, $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. $\boldsymbol{R}^+$ denotes the set of positive real numbers. The notation $\boldsymbol{X} > \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$), where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are symmetric matrices, means that $\boldsymbol{X} - \boldsymbol{Y}$ is positive definite (positive semi-definite). For a real matrix $\boldsymbol{B}$ and two real symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ of appropriate dimensions, $\begin{bmatrix} \boldsymbol{A} & * \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry. $\mathcal{E}\{x\}$ denotes the expectation of the stochastic variable $x$ and $\mathcal{E}\{x|y\}$ denotes the expectation of $x$ conditional on $y$.

## 2 Problem formulation and preliminaries

The considered system is given by the following mathematical model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}x(t) + \boldsymbol{A}_{\mathrm{d}}x(t - \tau(t)) + \boldsymbol{B}w(t) \tag{2}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}x(t) + \boldsymbol{C}_{\mathrm{d}}x(t - \tau(t)) + \boldsymbol{C}w(t) \tag{3}$$

$$\boldsymbol{z}(t) = \boldsymbol{L}x(t) + \boldsymbol{L}_{\mathrm{d}}x(t - \tau(t)) + \boldsymbol{L}_w w(t)x \tag{4}$$

$$\boldsymbol{x}(t) = \psi(t), \quad t \in [-\tau_2, 0] \tag{5}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{y}(t) \in \mathbb{R}^m$ and $\boldsymbol{z}(t) \in \mathbb{R}^q$ are the state vector, measurement vector and the signal vector to be estimated, respectively; $w(t)$ is the disturbance input; $\boldsymbol{A}$, $\boldsymbol{A}_{\mathrm{d}}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{C}_{\mathrm{d}}$, $\boldsymbol{D}$, $\boldsymbol{L}$, $\boldsymbol{L}_{\mathrm{d}}$ and $\boldsymbol{L}_w$ are the parameter matrices with appropriate dimensions. $\tau(t) \in [0,\ \tau_2]$ is the time-varying delay with an upper bound of $\tau_2$.

*Assumption 1:* $\tau(t)$ changes randomly and for a constant $\tau_1 \in [0, \tau_2]$, the probability of $\tau(t) \in [0, \tau_1)$ and $\tau(t) \in [\tau_1, \tau_2]$ can be known.

Define two sets

$$\boldsymbol{\Omega}_1 = \{t\colon \tau(t) \in [0, \tau_1)\} \tag{6}$$

$$\boldsymbol{\Omega}_2 = \{t\colon \tau(t) \in [\tau_1, \tau_2]\} \tag{7}$$

Furthermore, define two functions as

$$\tau_1(t) = \begin{cases} \tau(t), & \text{for } t \in \boldsymbol{\Omega}_1 \\ 0, & \text{for } t \notin \boldsymbol{\Omega}_1 \end{cases} \tag{8}$$

$$\tau_2(t) = \begin{cases} \tau(t), & \text{for } t \in \boldsymbol{\Omega}_2 \\ \tau_1, & \text{for } t \notin \boldsymbol{\Omega}_2 \end{cases} \tag{9}$$

Obviously, $\boldsymbol{\Omega}_1 \cup \boldsymbol{\Omega}_2 = \boldsymbol{R}^+$ and $\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2 = \Phi$ (empty set).

From the definitions of $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$, it can be seen that $t \in \boldsymbol{\Omega}_1$ means the event $\tau(t) \in [0,\ \tau_1)$ occurs and $t \in \boldsymbol{\Omega}_2$ means the event $\tau(t) \in [\tau_1,\ \tau_2]$ occurs. Corresponding to $\tau(t)$ taking values in different intervals, a stochastic variable $\beta(t)$ is defined as

$$\beta(t) = \begin{cases} 1, & t \in \boldsymbol{\Omega}_1 \\ 0, & t \in \boldsymbol{\Omega}_2 \end{cases} \tag{10}$$

*Assumption 2:* $\beta(t)$ is a Bernoulli distributed sequence with

$$\begin{aligned} &\mathrm{Prob}\{\beta(t) = 1\} = \mathcal{E}\{\beta(t)\} = \beta_0, \\ &\quad \mathrm{Prob}\{\beta(t) = 0\} = 1 - \mathcal{E}\{\beta(t)\} = 1 - \beta_0 \end{aligned} \tag{11}$$

where $0 \leq \beta_0 \leq 1$ is a constant.

*Remark 1:* The introduction of $\beta(t)$ is motivated by [25–27], where the Bernoulli distributed sequence $\beta(t)$ is used to model the missing message of the systems. Different from [25–27], $\beta(t)$ is used in this paper to describe the time-varying delay taking values in different intervals.

*Remark 2:* From Assumption 2, it can be shown that $\mathcal{E}\{\beta(t) - \beta_0\} = 0$ and $\mathcal{E}\{(\beta(t) - \beta_0)^2\} = \beta_0(1 - \beta_0)$. Since $\mathrm{Prob}\{\tau(t) \in [0, \tau_1)\} = \mathrm{Prob}\{\beta(t) = 1\}$ and $\mathrm{Prob}\{\tau(t) \in [\tau_1, \infty)\} = \mathrm{Prob}\{\beta(t) = 0\}$, $\beta_0$ and $1 - \beta_0$, respectively, denote the probability of $\tau(t)$ taking values in $[0, \tau_1)$ and $[\tau_1, \tau_2]$.

By using the new functions $\tau_i(t)$ $(i = 1, 2)$ and $\beta(t)$, (2)–(5) can be rewritten as

$$\begin{aligned} \dot{x}(t) = {} & \boldsymbol{A}x(t) + \beta(t)\boldsymbol{A}_{\mathrm{d}}x(t - \tau_1(t)) \\ & + (1 - \beta(t))\boldsymbol{A}_{\mathrm{d}}x(t - \tau_2(t)) + \boldsymbol{B}w(t) \end{aligned} \tag{12}$$

$$\begin{aligned} y(t) = {} & \boldsymbol{C}x(t) + \beta(t)\boldsymbol{C}_{\mathrm{d}}x(t - \tau_1(t)) \\ & + (1 - \beta(t))\boldsymbol{C}_{\mathrm{d}}x(t - \tau_2(t)) + \boldsymbol{D}w(t) \end{aligned} \tag{13}$$

$$\begin{aligned} z(t) = {} & \boldsymbol{L}x(t) + \beta(t)\boldsymbol{L}_{\mathrm{d}}x(t - \tau_1(t)) \\ & + (1 - \beta(t))\boldsymbol{L}_{\mathrm{d}}x(t - \tau_2(t)) + \boldsymbol{L}_w w(t) \end{aligned} \tag{14}$$

$$x(t) = \psi(t), \quad t \in [-\tau_2, 0] \tag{15}$$

The purpose of this paper is to construct a filter that can be used to estimate the signal $z(t)$. The structure of the designed filter has the following form

$$\dot{x}_{\mathrm{f}}(t) = \boldsymbol{A}_{\mathrm{f}}x_{\mathrm{f}}(t) + \boldsymbol{B}_{\mathrm{f}}y(t) \tag{16}$$

$$z_{\mathrm{f}}(t) = \boldsymbol{C}_{\mathrm{f}}x_{\mathrm{f}}(t) + \boldsymbol{D}_{\mathrm{f}}y(t) \tag{17}$$

where $\boldsymbol{A}_{\mathrm{f}}$, $\boldsymbol{B}_{\mathrm{f}}$, $\boldsymbol{C}_{\mathrm{f}}$ and $\boldsymbol{D}_{\mathrm{f}}$ are the filter parameters of appropriate dimensions to be determined.

Define $\eta(t) = \begin{bmatrix} x(t) \\ x_{\mathrm{f}}(t) \end{bmatrix}$, and $\mathrm{e}(t) = z(t) - z_{\mathrm{f}}(t)$. Combining (12)–(14), (16) and (17), we can obtain the filtering-error



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system

$$\dot{\eta}(t) = \tilde{A}\eta(t) + \beta(t)\tilde{A}_{\mathrm{d}}x(t-\tau_1(t)) + (1-\beta(t))\tilde{A}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{B}w(t) \quad (18)$$

$$\mathrm{e}(t) = \tilde{C}\eta(t) + \beta(t)\tilde{C}_{\mathrm{d}}x(t-\tau_1(t)) + (1-\beta(t))\tilde{C}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{D}w(t) \quad (19)$$

where

$$\tilde{A} = \begin{bmatrix} \boldsymbol{A} & 0 \\ \boldsymbol{B}_{\mathrm{f}}\boldsymbol{C} & \boldsymbol{A}_{\mathrm{f}} \end{bmatrix}, \quad \tilde{A}_{\mathrm{d}} = \begin{bmatrix} \boldsymbol{A}_{\mathrm{d}} \\ \boldsymbol{B}_{\mathrm{f}}\boldsymbol{C}_{\mathrm{d}} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B}_{\mathrm{f}}\boldsymbol{D} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \boldsymbol{L} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{C} & -\boldsymbol{C}_{\mathrm{f}} \end{bmatrix}, \quad \tilde{C}_{\mathrm{d}} = \boldsymbol{L}_{\mathrm{d}} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{C}_{\mathrm{d}}$$

$$\tilde{D} = \boldsymbol{L}_w - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{D}$$

Before giving the main result, we need the following definitions and lemma.

*Definition 1:* The system (2)–(4) is said to be exponentially stable in the mean-square sense (ESMSS), if there exist constants $\alpha > 0$, $\lambda > 0$, such that $t > 0$

$$\mathcal{E}\{\| x(t) \|^2\} \le \alpha \mathrm{e}^{-\lambda t} \sup_{-\tau_M < s < 0} \{\| \phi(s) \|^2\} \quad (20)$$

*Definition 2:* System (18) and (19) is said to be asymptotically stable with an $H_\infty$ norm bound $\gamma$, if the following conditions hold:

- The filtering-error system (18) and (19) with $w(t) = 0$ is asymptotically stable.
- Under the assumption of zero initial condition, the controlled output $\mathrm{e}(t)$ satisfies $\|\mathrm{e}(t)\|_2 \le \gamma \|w(t)\|_2$ for any non-zero $w(t) \in \mathcal{L}_2[t_0, \infty)$.

Since the augmented system (12)–(14) contains the stochastic quantity (i.e. $\beta(t)$), we need to introduce the notion of stochastic stability in the mean-square sense and the infinitesimal operator $\mathcal{L}(\cdot)$.

*Definition 3:* For a given function $\boldsymbol{V}: \boldsymbol{C}^b_{F_0}([-\tau_2, \quad 0], \boldsymbol{R}^n) \times \boldsymbol{S}$, its infinitesimal operator $\boldsymbol{L}\}$ is defined as

$$\mathcal{L}\boldsymbol{V}(\eta(t)) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}[\mathcal{E}(\boldsymbol{V}(\eta_{t+\Delta})|\eta_t) - \boldsymbol{V}(\eta_t)] \quad (21)$$

With Definition 1, our purpose is to develop filters of the filtering-error system (18) and (19) such that

1. The filtering-error systems (18) and (19) are asymptotically stable in the mean-square sense.
2. The filtering-error systems (18) and (19) guarantee a noise attenuation level in an $H_\infty$ sense, that is, under the assumption of zero initial condition for all non-zero $w(t) \in [0, \infty)$, it should be guaranteed that $\|e(t)\|_2 \le \gamma \|w(t)\|_2$.

The following Lemma makes an effective way to reduce the conservative of the result.

*Lemma 1 [28]:* $\boldsymbol{\Xi}_{1i}$, $\boldsymbol{\Xi}_{2i}$ $(i = 1, 2)$ and $\boldsymbol{\Omega}$ are constant matrices of appropriate dimensions, $\tau_i(t)$ is function of $t$ and satisfies $0 \le \tau_1(t) \le \tau_1 \le \tau_2(t) \le \tau_2$, $i = 1, 2$, then

$$[\tau_1(t)\boldsymbol{\Xi}_{11} + (\tau_1 - \tau_1(t))\boldsymbol{\Xi}_{21}] + [(\tau_2(t) - \tau_1)\boldsymbol{\Xi}_{12} + (\tau_2 - \tau_2(t))\boldsymbol{\Xi}_{22}] + \boldsymbol{\Omega} < 0 \quad (22)$$

if and only if the following four inequalities hold

$$\tau_1\boldsymbol{\Xi}_{11} + (\tau_2 - \tau_1)\boldsymbol{\Xi}_{12} + \boldsymbol{\Omega} < 0 \quad (23)$$

$$\tau_1\boldsymbol{\Xi}_{11} + (\tau_2 - \tau_1)\boldsymbol{\Xi}_{22} + \boldsymbol{\Omega} < 0 \quad (24)$$

$$\tau_1\boldsymbol{\Xi}_{21} + (\tau_2 - \tau_1)\boldsymbol{\Xi}_{12} + \boldsymbol{\Omega} < 0 \quad (25)$$

$$\tau_1\boldsymbol{\Xi}_{21} + (\tau_2 - \tau_1)\boldsymbol{\Xi}_{22} + \boldsymbol{\Omega} < 0 \quad (26)$$

## 3 Main results

The following result can be obtained for systems (18) and (19).

*Theorem 1:* For some given constants $0 \le \tau_1 \le \tau_2$ and $\gamma$, the systems (18) and (19) is ESMSS if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q}_1 > 0$, $\boldsymbol{Q}_2 > 0$, $\boldsymbol{R}_1 > 0$, $\boldsymbol{R}_2 > 0$ and matrices $\boldsymbol{N}_i$, $\boldsymbol{M}_i$, $\boldsymbol{V}_i$, $\boldsymbol{S}_i (i = 1, 2, \ldots, 5)$ of appropriate dimensions such that

$$\boldsymbol{\Xi}(l) = \begin{bmatrix} \boldsymbol{\Xi}_{11} + \boldsymbol{\Omega} + \boldsymbol{\Omega}^{\mathrm{T}} & * & * & * \\ \boldsymbol{\Xi}_{21} & \boldsymbol{\Xi}_{22} & * & * \\ \boldsymbol{\Xi}_{31} & \boldsymbol{\Xi}_{32} & \boldsymbol{\Xi}_{33} & * \\ \boldsymbol{\Xi}^{(l)}_{41} & 0 & 0 & \boldsymbol{\Xi}_{44} \end{bmatrix} < 0,$$

$$l = 1, 2, 3, 4 \quad (27)$$

where

$$\boldsymbol{\Xi}_{11} = \begin{bmatrix} \Gamma_1 & * & * & * & * \\ \beta_0\tilde{A}^{\mathrm{T}}_{\mathrm{d}}\boldsymbol{P} & 0 & * & * & * \\ 0 & 0 & -\boldsymbol{Q}_1 & * & * \\ (1-\beta_0)\tilde{A}^{\mathrm{T}}_{\mathrm{d}}\boldsymbol{P} & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & -\boldsymbol{Q}_2 \end{bmatrix}$$

$$\boldsymbol{\Xi}_{21} = \begin{bmatrix} \tilde{B}^{\mathrm{T}}_1\mathcal{P} \\ \sqrt{\beta_0}\boldsymbol{L}_0 \\ \sqrt{(1-\beta_0)}\boldsymbol{L}_1 \end{bmatrix}, \quad \boldsymbol{\Xi}_{22} = \begin{bmatrix} -\gamma^2\boldsymbol{I} & * & * \\ \sqrt{\beta_0}\tilde{D} & -\boldsymbol{I} & * \\ \sqrt{1-\beta_0}\tilde{D} & 0 & -\boldsymbol{I} \end{bmatrix}$$

$$\boldsymbol{\Xi}_{31} = \begin{bmatrix} \sqrt{\tau_1\beta_0}\boldsymbol{R}_1 H\mathcal{A}_0 \\ \sqrt{\tau_1(1-\beta_0)}\boldsymbol{R}_1 H\mathcal{A}_1 \\ \sqrt{(\tau_2-\tau_1)\beta_0}\boldsymbol{R}_2 H\mathcal{A}_0 \\ \sqrt{(\tau_2-\tau_1)(1-\beta_0)}\boldsymbol{R}_2 H\mathcal{A}_1 \end{bmatrix}$$

$$\boldsymbol{\Xi}_{32} = \begin{bmatrix} \sqrt{\tau_1\beta_0}\boldsymbol{R}_1 H\mathcal{B} \\ \sqrt{\tau_1(1-\beta_0)}\boldsymbol{R}_1 H\mathcal{B} \\ \sqrt{(\tau_2-\tau_1)\beta_0}\boldsymbol{R}_2 H\mathcal{B} \\ \sqrt{(\tau_2-\tau_1)(1-\beta_0)}\boldsymbol{R}_2 H\mathcal{B} \end{bmatrix}$$

$$\Xi_{33} = \mathrm{diag}\begin{pmatrix}-\boldsymbol{R}_1 & -\boldsymbol{R}_1 & -\boldsymbol{R}_2 & -\boldsymbol{R}_2\end{pmatrix}$$
$$\Xi_{44} = \mathrm{diag}\begin{pmatrix}-\boldsymbol{R}_1 & -\boldsymbol{R}_2\end{pmatrix}$$
$$\Xi_{41}^{(1)} = \begin{bmatrix}\sqrt{\tau_1}\boldsymbol{N}^{\mathrm{T}} \\ \sqrt{\tau_2-\tau_1}\boldsymbol{V}^{\mathrm{T}}\end{bmatrix}, \quad \Xi_{41}^{(2)} = \begin{bmatrix}\sqrt{\tau_1}\boldsymbol{N}^{\mathrm{T}} \\ \sqrt{\tau_2-\tau_1}\boldsymbol{S}^{\mathrm{T}}\end{bmatrix}$$
$$\Xi_{41}^{(3)} = \begin{bmatrix}\sqrt{\tau_1}\boldsymbol{M}^{\mathrm{T}} \\ \sqrt{\tau_2-\tau_1}\boldsymbol{V}^{\mathrm{T}}\end{bmatrix}, \quad \Xi_{41}^{(4)} = \begin{bmatrix}\sqrt{\tau_1}\boldsymbol{M}^{\mathrm{T}} \\ \sqrt{\tau_2-\tau_1}\boldsymbol{S}^{\mathrm{T}}\end{bmatrix}$$

$$\mathcal{A}_0 = \begin{bmatrix}\tilde{A} & \tilde{A}_{\mathrm{d}} & 0 & 0 & 0\end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix}\tilde{A} & 0 & 0 & \tilde{A}_{\mathrm{d}} & 0\end{bmatrix}$$
$$\mathcal{B} = \begin{bmatrix}\tilde{B} & 0 & 0\end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix}\boldsymbol{P} & 0 & 0 & 0 & 0\end{bmatrix}$$
$$\boldsymbol{L}_0 = \begin{bmatrix}\tilde{C} & \tilde{C}_{\mathrm{d}} & 0 & 0 & 0\end{bmatrix}, \quad \boldsymbol{L}_1 = \begin{bmatrix}\tilde{C} & 0 & 0 & \tilde{C}_{\mathrm{d}} & 0\end{bmatrix}$$
$$\Gamma_1 = \boldsymbol{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\boldsymbol{P} + H^{\mathrm{T}}(\boldsymbol{Q}_1 + \boldsymbol{Q}_2)H + \boldsymbol{N}_1 H + H^{\mathrm{T}}\boldsymbol{N}_1$$
$$\boldsymbol{\Omega} = \begin{bmatrix}\boldsymbol{N}H & -\boldsymbol{N}+\boldsymbol{M} & -\boldsymbol{M}+\boldsymbol{V} & -\boldsymbol{V}+\boldsymbol{S} & -\boldsymbol{S}\end{bmatrix}$$
$$H = \begin{bmatrix}\boldsymbol{I} & 0\end{bmatrix}$$
$$\boldsymbol{N}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{N}_1^{\mathrm{T}} & \boldsymbol{N}_2^{\mathrm{T}} & \boldsymbol{N}_3^{\mathrm{T}} & \boldsymbol{N}_4^{\mathrm{T}} & \boldsymbol{N}_5^{\mathrm{T}}\end{bmatrix}$$
$$\boldsymbol{M}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{M}_1^{\mathrm{T}} & \boldsymbol{M}_2^{\mathrm{T}} & \boldsymbol{M}_3^{\mathrm{T}} & \boldsymbol{M}_4^{\mathrm{T}} & \boldsymbol{M}_5^{\mathrm{T}}\end{bmatrix}$$
$$\boldsymbol{V}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{V}_1^{\mathrm{T}} & \boldsymbol{V}_2^{\mathrm{T}} & \boldsymbol{V}_3^{\mathrm{T}} & \boldsymbol{V}_4^{\mathrm{T}} & \boldsymbol{V}_5^{\mathrm{T}}\end{bmatrix}$$
$$\boldsymbol{S}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{S}_1^{\mathrm{T}} & \boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S}_3^{\mathrm{T}} & \boldsymbol{S}_4^{\mathrm{T}} & \boldsymbol{S}_5^{\mathrm{T}}\end{bmatrix}$$

*Proof:* The Lyapunov–Krasovskii functional candidate is chosen as

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{28}$$

where

$$V_1(x_t) = \eta^{\mathrm{T}}(t)\boldsymbol{P}\eta(t)$$

$$V_2(x_t) = \int_{t-\tau_1}^{t} x^{\mathrm{T}}(s)\boldsymbol{Q}_1 x(s)\,\mathrm{d}s + \int_{t-\tau_2}^{t} x^{\mathrm{T}}(s)\boldsymbol{Q}_2 x(s)\,\mathrm{d}s$$

$$V_3(x_t) = \int_{t-\tau_1}^{t}\int_{s}^{t} \dot{x}^{\mathrm{T}}(v)\boldsymbol{R}_1\dot{x}(v)\,\mathrm{d}v\,\mathrm{d}s + \int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t} \dot{x}^{\mathrm{T}}(v)\boldsymbol{R}_2\dot{x}(v)\,\mathrm{d}v\,\mathrm{d}s$$

Using the infinitesimal operator (21), we obtain

$$\mathcal{L}V_1(x_t) = 2\eta^{\mathrm{T}}(t)\boldsymbol{P}(\tilde{A}\eta(t) + \beta_0\tilde{A}_{\mathrm{d}}x(t-\tau_1(t)) + (1-\beta_0)\tilde{A}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{B}w(t)) \tag{29}$$

$$\mathcal{L}V_2(x_t) = x^{\mathrm{T}}(t)(\boldsymbol{Q}_1+\boldsymbol{Q}_2)x(t) - x^{\mathrm{T}}(t-\tau_1)\boldsymbol{Q}_1 x(t-\tau_1) - x^{\mathrm{T}}(t-\tau_2)\boldsymbol{Q}_2 x(t-\tau_2) \tag{30}$$

$$\mathcal{L}V_3(x_t) = \dot{x}^{\mathrm{T}}(t)(\tau_1\boldsymbol{R}_1 + (\tau_2-\tau_1)\boldsymbol{R}_2)\dot{x}(t) - \int_{t-\tau_1}^{t}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{x}(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{x}(s)\,\mathrm{d}s \tag{31}$$

By employing the free matrix method [29, 30], we obtain

$$\begin{aligned}\mathcal{L}V(x_t) = {} & 2\eta^{\mathrm{T}}(t)\boldsymbol{P}(\tilde{A}\eta(t) + \beta_0\tilde{A}_{\mathrm{d}}x(t-\tau_1(t)) \\ & + (1-\beta_0)\tilde{A}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{B}w(t)) \\ & + x^{\mathrm{T}}(t)(\boldsymbol{Q}_1+\boldsymbol{Q}_2)x(t) - x^{\mathrm{T}}(t-\tau_1)\boldsymbol{Q}_1x(t-\tau_1) \\ & - x^{\mathrm{T}}(t-\tau_2)\boldsymbol{Q}_2x(t-\tau_2) + \dot{x}^{\mathrm{T}}(t)(\tau_1\boldsymbol{R}_1 \\ & + (\tau_2-\tau_1)\boldsymbol{R}_2)\dot{x}(t) - \int_{t-\tau_1}^{t}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{x}(s)\,\mathrm{d}s \\ & - \int_{t-\tau_2}^{t-\tau_1}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{x}(s)\,\mathrm{d}s + 2\zeta^{\mathrm{T}}(t) \\ & \times \boldsymbol{N}\left[H\eta(t) - x(t-\tau_1(t)) - \int_{t-\tau_1(t)}^{t}\dot{x}(s)\,\mathrm{d}s\right] \\ & + 2\zeta^{\mathrm{T}}(t)\boldsymbol{M}\left[x(t-\tau_1(t)) - x(t-\tau_1) - \int_{t-\tau_1}^{t-\tau_1(t)}\dot{x}(s)\,\mathrm{d}s\right] \\ & + 2\zeta^{\mathrm{T}}(t)\boldsymbol{V}\left[x(t-\tau_1) - x(t-\tau_2(t)) - \int_{t-\tau_2(t)}^{t-\tau_1}\dot{x}(s)\,\mathrm{d}s\right] \\ & + 2\zeta^{\mathrm{T}}(t)\boldsymbol{S}\left[x(t-\tau_2(t)) - x(t-\tau_2) - \int_{t-\tau_2}^{t-\tau_2(t)}\dot{x}(s)\,\mathrm{d}s\right] \\ & + \mathrm{e}^{\mathrm{T}}(t)\mathrm{e}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) - \mathrm{e}^{\mathrm{T}}(t)\mathrm{e}(t) + \gamma^2 w^{\mathrm{T}}(t)w(t)\end{aligned} \tag{32}$$

where

$$\zeta^{\mathrm{T}}(t) = \begin{bmatrix}\eta^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau_1(t)) & x^{\mathrm{T}}(t-\tau_1) & x^{\mathrm{T}}(t-\tau_2(t)) & x^{\mathrm{T}}(t-\tau_2)\end{bmatrix}$$

Note that

$$\dot{\eta}(t) = \beta(t)[\tilde{A}\eta(t) + \tilde{A}_{\mathrm{d}}x(t-\tau_1(t)) + \tilde{B}w(t)] + (1-\beta(t))[\tilde{A}\eta(t) + \tilde{A}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{B}w(t)] \tag{33}$$

$$\mathrm{e}(t) = \beta(t)[\tilde{C}\eta(t) + \tilde{C}_{\mathrm{d}}x(t-\tau_1(t)) + \tilde{D}w(t)] + (1-\beta(t))[\tilde{C}\eta(t) + \tilde{C}_{\mathrm{d}}x(t-\tau_2(t)) + \tilde{D}w(t)] \tag{34}$$

From (33) and (34), we obtain

$$\mathcal{E}\{\mathrm{e}^{\mathrm{T}}(t)\mathrm{e}(t)\} = \beta_0\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{L}_0^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{L}_0 & \tilde{D}\end{bmatrix}\bar{\zeta}(t) + (1-\beta_0)\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{L}_1^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{L}_1 & \tilde{D}\end{bmatrix}\bar{\zeta}(t) \tag{35}$$




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$$\begin{aligned}
&\mathcal{E}\{\dot{x}^{\mathrm{T}}(t)[\tau_1\boldsymbol{R}_1+(\tau_2-\tau_1)\boldsymbol{R}_2]\dot{x}(t)\}\\
&=\mathcal{E}\{\dot{\eta}^{\mathrm{T}}(t)H^{\mathrm{T}}[\tau_1\boldsymbol{R}_1+(\tau_2-\tau_1)\boldsymbol{R}_2]H\dot{\eta}(t)\}\\
&=\beta_0\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\mathcal{A}_0^{\mathrm{T}}\\ \tilde{B}^{\mathrm{T}}\end{bmatrix}H^{\mathrm{T}}[\tau_1\boldsymbol{R}_1+(\tau_2-\tau_1)\boldsymbol{R}_2]H\begin{bmatrix}\mathcal{A}_0 & \tilde{B}\end{bmatrix}\bar{\zeta}(t)\\
&\quad+(1-\beta_0)\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\mathcal{A}_1^{\mathrm{T}}\\ \tilde{B}^{\mathrm{T}}\end{bmatrix}H^{\mathrm{T}}[\tau_1\boldsymbol{R}_1+(\tau_2-\tau_1)\boldsymbol{R}_2]H\begin{bmatrix}\mathcal{A}_1 & \tilde{B}\end{bmatrix}\bar{\zeta}(t)
\end{aligned}\tag{36}$$

where $\bar{\zeta}^{\mathrm{T}}(t)=\begin{bmatrix}\zeta^{\mathrm{T}}(t) & w^{\mathrm{T}}(t)\end{bmatrix}$.

From (32), (35) and (36), we can obtain

$$\begin{aligned}
&\mathcal{L}\boldsymbol{V}(x_t)+\mathrm{e}^{\mathrm{T}}(t)\mathrm{e}(t)-\gamma^2w^{\mathrm{T}}(t)w(t)\\
&\le\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{\Xi}_{11}+\boldsymbol{\Omega}+\boldsymbol{\Omega}^{\mathrm{T}} & * & *\\ \boldsymbol{\Xi}_{21} & \boldsymbol{\Xi}_{22} & *\\ \boldsymbol{\Xi}_{31} & \boldsymbol{\Xi}_{32} & \boldsymbol{\Xi}_{33}\end{bmatrix}\bar{\zeta}(t)\\
&\quad+\tau_1(t)\zeta^{\mathrm{T}}(t)\boldsymbol{N}\boldsymbol{R}_1^{-1}\boldsymbol{N}^{\mathrm{T}}\zeta(t)+[\tau_1-\tau_1(t)]\zeta^{\mathrm{T}}(t)\boldsymbol{M}\boldsymbol{R}_1^{-1}\boldsymbol{M}^{\mathrm{T}}\zeta(t)\\
&\quad+[\tau_2(t)-\tau_1]\zeta^{\mathrm{T}}(t)\boldsymbol{V}\boldsymbol{R}_2^{-1}\boldsymbol{V}^{\mathrm{T}}\zeta(t)\\
&\quad+[\tau_2-\tau_2(t)]\zeta^{\mathrm{T}}(t)\boldsymbol{S}\boldsymbol{R}_2^{-1}\boldsymbol{S}^{\mathrm{T}}\zeta(t)
\end{aligned}\tag{37}$$

Using Lemma 1, it is easy to see that the condition (27) is a sufficient condition to guarantee

$$\begin{aligned}
&\bar{\zeta}^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{\Xi}_{11}+\boldsymbol{\Omega}+\boldsymbol{\Omega}^{\mathrm{T}} & * & *\\ \boldsymbol{\Xi}_{21} & \boldsymbol{\Xi}_{22} & *\\ \boldsymbol{\Xi}_{31} & \boldsymbol{\Xi}_{32} & \boldsymbol{\Xi}_{33}\end{bmatrix}\bar{\zeta}(t)\\
&\quad+\tau_1(t)\zeta^{\mathrm{T}}(t)\boldsymbol{N}\boldsymbol{R}_1^{-1}\boldsymbol{N}^{\mathrm{T}}\zeta(t)\\
&\quad+[\tau_1-\tau_1(t)]\zeta^{\mathrm{T}}(t)\boldsymbol{M}\boldsymbol{R}_1^{-1}\boldsymbol{M}^{\mathrm{T}}\zeta(t)\\
&\quad+[\tau_2(t)-\tau_1]\zeta^{\mathrm{T}}(t)\boldsymbol{V}\boldsymbol{R}_2^{-1}\boldsymbol{V}^{\mathrm{T}}\zeta(t)\\
&\quad+[\tau_2-\tau_2(t)]\zeta^{\mathrm{T}}(t)\boldsymbol{S}\boldsymbol{R}_2^{-1}\boldsymbol{S}^{\mathrm{T}}\zeta(t)<0
\end{aligned}\tag{38}$$

By Schur complement, it can be concluded from (37) that

$$\mathcal{L}\boldsymbol{V}(x_t)\le-\mathrm{e}^{\mathrm{T}}(t)\mathrm{e}(t)+\gamma^2\omega^{\mathrm{T}}(t)\omega(t)\tag{39}$$

Integrating both sides of (39) from 0 to $t$ yields

$$\mathcal{E}\{\boldsymbol{V}(t)\}-\mathcal{E}\{\boldsymbol{V}(0)\}\le-\int_0^t\mathrm{e}^{\mathrm{T}}(s)\mathrm{e}(s)\,\mathrm{d}s+\int_0^t\gamma^2\omega^{\mathrm{T}}(s)\omega(s)\,\mathrm{d}s\tag{40}$$

Then, letting $t\to\infty$ and under zero initial condition, we can show from (40) that

$$\int_0^t\mathrm{e}^{\mathrm{T}}(s)\mathrm{e}(s)\,\mathrm{d}s\le\int_0^t\gamma^2\omega^{\mathrm{T}}(s)\omega(s)\,\mathrm{d}s\tag{41}$$

thus, $\|\mathrm{e}(t)\|_2\le\gamma\le\|\omega(t)\|_2$.

Next, we prove the ESMSS of the systems (18) and (19). In this situation, the external perturbation $\omega(t)$ is assumed to be zero. Then, similar to the above analysis, we can conclude that

$$\mathcal{E}\{\mathcal{L}\boldsymbol{V}(x_t)\}\le-\lambda||\zeta(t)||$$

where $\lambda=\lambda_{\min}\{\boldsymbol{\Xi}(l)\}(l=1,2,3,4)$. Define a new function as

$$W(x_t)=\mathrm{e}^{\varepsilon t}\boldsymbol{V}(x_t)\tag{42}$$

Its infinitesimal operator $\mathcal{L}$ is given by

$$\mathcal{L}W(x_t)=\varepsilon\mathrm{e}^{\varepsilon t}\boldsymbol{V}(x_t)+\mathrm{e}^{\varepsilon t}\mathcal{L}\boldsymbol{V}(x_t)\tag{43}$$

Then, we can obtain from (43) that

$$\begin{aligned}
\mathcal{E}\{W(x_t)\}-\mathcal{E}\{W(x_0)\}&=\int_0^t\varepsilon\mathrm{e}^{\varepsilon s}\mathcal{E}\{\boldsymbol{V}(x_s)\}\,\mathrm{d}s\\
&\quad+\int_0^t\mathrm{e}^{\varepsilon s}\mathcal{E}\{\mathcal{L}\boldsymbol{V}(x_s)\}\,\mathrm{d}s
\end{aligned}\tag{44}$$

Using the similar method of [31], we can see that there exists a positive number $\alpha$ such that for $t\ge0$

$$\mathcal{E}\{\boldsymbol{V}(x_t)\}\le\alpha\sup_{-2\tau_2\le s\le0}\mathcal{E}\{||\tilde{\psi}(s)||^2\}\mathrm{e}^{-\varepsilon t}\tag{45}$$

Since $\boldsymbol{V}(x_t)\ge\{\lambda_{\min}\le(\boldsymbol{P})\}\eta^{\mathrm{T}}(t)\eta(t)$, it can be shown from (45) that for $t\ge0$

$$\mathcal{E}\{\eta^{\mathrm{T}}(t)\eta(t)\}\le\bar{\alpha}\mathrm{e}^{-\varepsilon t}\sup_{-2\tau_2\le s\le0}\mathcal{E}\{||\tilde{\psi}(s)||^2\}\tag{46}$$

where $\bar{\alpha}=\{\alpha/(\lambda_{\min}(\boldsymbol{P}))\}$. Recalling Definition 1, the proof can be completed. □

*Remark 3:* Using the methods in [19, 22], the time-varying delay $\tau(t)$ often appears in the derivation of the Lyapunov functional or the introduced free-weighing matrix equations, such as $\int_{t-\tau(t)}^{t}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}\dot{x}(s)\,\mathrm{d}s$ and $\tau(t)\zeta^{\mathrm{T}}(t)\boldsymbol{X}\zeta(t)$ ($\boldsymbol{R}>0$ and $\boldsymbol{X}>0$), which is expanded to $\int_{t-\tau_2}^{t}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}\dot{x}(s)\,\mathrm{d}s$ and $\tau_2\zeta^{\mathrm{T}}(t)\boldsymbol{X}\zeta(t)$ by using the method in [19, 22], then the estimation errors $\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^{\mathrm{T}}(s)\boldsymbol{R}\dot{x}(s)\,\mathrm{d}s$ and $(\tau_2-\tau(t))\ \zeta^{\mathrm{T}}(t)\boldsymbol{X}\ \zeta(t)$ are ignored, which will unavoidably lead to some degree of conservativeness. However, from the proof of Theorem 1, it can be seen that there is no expansion for $\tau(t)$, therefore the conservatism caused by expanding $\tau(t)$ to $\tau_2$ can be avoided.

As a special case, we consider $\beta(t)\equiv1$, that is, the information of probability distribution is not taken into consideration, and in this case, the systems (18) and (19) reduce to the system

$$\dot{\eta}(t)=\tilde{A}\eta(t)+\tilde{A}_{\mathrm{d}}x(t-\tau(t))+\tilde{B}w(t)\tag{47}$$

$$\mathrm{e}(t)=\tilde{C}\eta(t)+\tilde{C}_{\mathrm{d}}x(t-\tau(t))+\tilde{D}w(t)\tag{48}$$

Similar to Theorem 1, the following result can be obtained.

*Corollary 1:* For some given constants $\tau_2$ and $\gamma$, the systems (47) and (48) are asymptotically stable if there exist matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}_2>0$, $\boldsymbol{R}_2>0$ and matrices $\boldsymbol{N}_i$, $\boldsymbol{S}_i(i=1,\ 2)$ of appropriate dimensions such that

$$\Psi=\begin{bmatrix}\Psi_{11} & * & *\\ \Psi_{21} & \Psi_{22} & *\\ \Psi_{31}^{(l)} & \Psi_{32}^{(l)} & -\boldsymbol{R}_2\end{bmatrix}<0,\ l=1,2\tag{49}$$

where (see equations at the bottom of the page)

*Proof:* Choose the Lyapunov functional as

$$V(x_t) = \eta^{\mathrm{T}}(t)\boldsymbol{P}\eta(t) + \int_{t-\tau_2}^{t} \eta^{\mathrm{T}}(s)H^{\mathrm{T}}\boldsymbol{Q}_2 H\eta(s)\,\mathrm{d}s + \int_{t-\tau_2}^{t}\int_{s}^{t} \dot{x}^{\mathrm{T}}(v)\boldsymbol{R}_2\dot{x}(v)\,\mathrm{d}v\,\mathrm{d}s \tag{50}$$

and the free-weighing matrix as

$$2\zeta^{\mathrm{T}}(t)\boldsymbol{N}\left[H\eta(t) - H\eta(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{x}(s)\,\mathrm{d}s\right] = 0 \tag{51}$$

$$2\zeta^{\mathrm{T}}(t)\boldsymbol{S}\left[x(t-\tau(t)) - x(t-\tau_2) - \int_{t-\tau_2}^{t-\tau(t)}\dot{x}(s)\,\mathrm{d}s\right] = 0 \tag{52}$$

where

$$\zeta^{\mathrm{T}}(t) = \begin{bmatrix} \eta^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau(t)) & x^{\mathrm{T}}(t-\tau_2) \end{bmatrix}$$

$$\boldsymbol{N}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{N}_1^{\mathrm{T}} & \boldsymbol{N}_2^{\mathrm{T}} & \boldsymbol{N}_3^{\mathrm{T}} \end{bmatrix} \tag{53}$$

$$\boldsymbol{S}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{S}_1^{\mathrm{T}} & \boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S}_3^{\mathrm{T}} \end{bmatrix} \tag{54}$$

Then, (49) can be obtained similar to the proof of Theorem 1.

In the inequality (27,49), matrices $\boldsymbol{P} > 0$ and filter parameters $\boldsymbol{A}_{\mathrm{f}}$, $\boldsymbol{B}_{\mathrm{f}}$, $\boldsymbol{C}_{\mathrm{f}}$ and $\boldsymbol{D}_{\mathrm{f}}$, that are included in the matrix $\tilde{A}$, $\tilde{A}_{\mathrm{d}}$, $\tilde{B}$, $\tilde{C}$, $\tilde{C}_{\mathrm{d}}$ and $\tilde{D}$ are unknown and occur in non-linear fashion. Hence, the inequality (27,49) cannot be considered as a linear matrix inequality (LMI) problem. In the following, a result is proposed to change variables such that the inequality can be solved.

*Theorem 2:* For some given constants $0 \le \tau_1 \le \tau_2$ and $\gamma$, the systems (18) and (19) are ESMSS if there exist matrices $\boldsymbol{P}_1 > 0$, $\bar{P}_3 > 0$, $\boldsymbol{Q}_i > 0$, $\boldsymbol{R}_i > 0$, $(i = 1,\ 2)$ and matrices $\bar{A}_f$, $\bar{B}_f$, $\bar{C}_f$, $\bar{D}_f$, $\boldsymbol{N}_{10}$, $\boldsymbol{N}_{11}$, $\boldsymbol{M}_{10}$, $\boldsymbol{M}_{11}$, $\boldsymbol{V}_{10}$, $\boldsymbol{V}_{11}$, $\boldsymbol{S}_{10}$, $\boldsymbol{S}_{11}$, $\boldsymbol{M}_i$, $\boldsymbol{N}_i$, $\boldsymbol{V}_i$, $\boldsymbol{S}_i (i = 2,\ 3,\ \ldots,\ 5)$, of appropriate dimensions such that the following LMIs hold

$$\begin{bmatrix} \Pi_{11} & * & * & * \\ \Pi_{21} & \Pi_{22} & * & * \\ \Pi_{31} & \Pi_{32} & \boldsymbol{\Xi}_{33} & * \\ \Pi_{41}^{(l)} & 0 & 0 & \boldsymbol{\Xi}_{44} \end{bmatrix} < 0, \quad l = 1, 2, 3, 4 \tag{55}$$

$$\boldsymbol{P}_1 - \bar{P}_3 > 0 \tag{56}$$

where (see equation at the bottom of the page)

$$\Pi_{22} = \begin{bmatrix} -\gamma^2\boldsymbol{I} & * & * \\ \sqrt{\beta_0}(\boldsymbol{L}_w - \bar{D}_{\mathrm{f}}\boldsymbol{D}) & -\boldsymbol{I} & * \\ \sqrt{1-\beta_0}(\boldsymbol{L}_w - \bar{D}_{\mathrm{f}}\boldsymbol{D}) & 0 & -\boldsymbol{I} \end{bmatrix}$$

(see equation at the bottom of the page)

$$\Pi_{32} = \begin{bmatrix} \sqrt{\beta_0\tau_1}\boldsymbol{R}_1\boldsymbol{B} & 0 & 0 \\ \sqrt{(1-\beta_0)\tau_1}\boldsymbol{R}_1\boldsymbol{B} & 0 & 0 \\ \sqrt{\beta_0(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{B} & 0 & 0 \\ \sqrt{(1-\beta_0)(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{B} & 0 & 0 \end{bmatrix}$$

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$$\Psi_{11} = \boldsymbol{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\boldsymbol{P} + H^{\mathrm{T}}\boldsymbol{Q}_2 H + \boldsymbol{N}_1 H + H^{\mathrm{T}}\boldsymbol{N}_1^{\mathrm{T}}$$

$$\Psi_{21} = \begin{bmatrix} \tilde{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{N}_2 H - \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{S}_1^{\mathrm{T}} \\ \boldsymbol{N}_3 H - \boldsymbol{S}_1^{\mathrm{T}} \\ \tilde{B}^{\mathrm{T}}\boldsymbol{P} \\ \sqrt{\tau_2}\boldsymbol{R}_2 H\tilde{A} \\ \tilde{C} \end{bmatrix}, \quad \Psi_{22} = \begin{bmatrix} \Sigma_2 & * & * & * & * \\ -\boldsymbol{N}_3 + \boldsymbol{S}_3 - \boldsymbol{S}_2^{\mathrm{T}} & -\boldsymbol{Q}_2 - \boldsymbol{S}_3 - \boldsymbol{S}_3^{\mathrm{T}} & * & * & * \\ 0 & 0 & -\gamma^2\boldsymbol{I} & * & * \\ \sqrt{\tau_2}\boldsymbol{R}_2\boldsymbol{A}_{\mathrm{d}} & 0 & \sqrt{\tau_2}\boldsymbol{R}_2\boldsymbol{B} & -\boldsymbol{R}_2 & * \\ \boldsymbol{L}_{\mathrm{d}} - \boldsymbol{D}_f\boldsymbol{C}_{\mathrm{d}} & 0 & \boldsymbol{L}_w - \boldsymbol{D}_f\boldsymbol{D} & 0 & -\boldsymbol{I} \end{bmatrix},$$

$$\Psi_{31}^{(1)} = \sqrt{\tau_2}\boldsymbol{N}_1^{\mathrm{T}}, \quad \Psi_{31}^{(2)} = \sqrt{\tau_2}\boldsymbol{S}_1^{\mathrm{T}}, \quad \Psi_{32}^{(1)} = \begin{bmatrix} \sqrt{\tau_2}\boldsymbol{N}_2^{\mathrm{T}} & \sqrt{\tau_2}\boldsymbol{N}_3^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_{32}^{(2)} = \begin{bmatrix} \sqrt{\tau_2}\boldsymbol{S}_2^{\mathrm{T}} & \sqrt{\tau_2}\boldsymbol{S}_3^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix},$$
$$\Sigma_2 = -\boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}} + \boldsymbol{S}_2 + \boldsymbol{S}_2^{\mathrm{T}}, \quad H = \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix}$$

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$$\Pi_{11} = \begin{bmatrix} \Lambda_1 & * & * & * & * & * \\ \Lambda_2 & \bar{A}_{\mathrm{f}} + \bar{A}_{\mathrm{f}}^{\mathrm{T}} & * & * & * & \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & * & * & * \\ \boldsymbol{N}_3 - \boldsymbol{M}_{10}^{\mathrm{T}} + \boldsymbol{V}_{10}^{\mathrm{T}} & -\boldsymbol{M}_{11}^{\mathrm{T}} + \boldsymbol{V}_{11}^{\mathrm{T}} & \Lambda_6 & \Lambda_7 & * & * \\ \Lambda_8 & \Lambda_9 & \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & * \\ \boldsymbol{N}_5 - \boldsymbol{S}_{10}^{\mathrm{T}} & -\boldsymbol{S}_{11}^{\mathrm{T}} & -\boldsymbol{N}_5 + \boldsymbol{M}_5 - \boldsymbol{S}_2^{\mathrm{T}} & -\boldsymbol{M}_5 + \boldsymbol{V}_5 - \boldsymbol{S}_3^{\mathrm{T}} & -\boldsymbol{V}_5 + \boldsymbol{S}_5 - \boldsymbol{S}_4^{\mathrm{T}} & -\boldsymbol{Q}_2 - \boldsymbol{S}_5 - \boldsymbol{S}_5^{\mathrm{T}} \end{bmatrix}$$

$$\Pi_{21} = \begin{bmatrix} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{D}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} & \boldsymbol{B}^{\mathrm{T}}\bar{P}_3 + \boldsymbol{D}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ \sqrt{\beta_0}(\boldsymbol{L} - \bar{D}_{\mathrm{f}}\boldsymbol{C}) & -\sqrt{\beta_0}\bar{C}_{\mathrm{f}} & \sqrt{\beta_0}(\boldsymbol{L}_{\mathrm{d}} - \bar{D}_{\mathrm{f}}\boldsymbol{C}_{\mathrm{d}}) & 0 & 0 & 0 \\ \sqrt{1-\beta_0}(\boldsymbol{L} - \bar{D}_{\mathrm{f}}\boldsymbol{C}) & -\sqrt{1-\beta_0}\bar{C}_{\mathrm{f}} & 0 & 0 & \sqrt{1-\beta_0}(\boldsymbol{L}_{\mathrm{d}} - \bar{D}_{\mathrm{f}}\boldsymbol{C}_{\mathrm{d}}) & 0 \end{bmatrix}$$

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$$\Pi_{31} = \begin{bmatrix} \sqrt{\beta_0\tau_1}\boldsymbol{R}_1\boldsymbol{A} & 0 & \sqrt{\beta_0\tau_1}\boldsymbol{R}_1\boldsymbol{A}_{\mathrm{d}} & 0 & 0 & 0 \\ \sqrt{(1-\beta_0)\tau_1}\boldsymbol{R}_1\boldsymbol{A} & 0 & 0 & 0 & \sqrt{(1-\beta_0)\tau_1}\boldsymbol{R}_1\boldsymbol{A}_{\mathrm{d}} & 0 \\ \sqrt{\beta_0(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{A} & 0 & \sqrt{\beta_0(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{A}_{\mathrm{d}} & 0 & 0 & 0 \\ \sqrt{(1-\beta_0)(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{A} & 0 & 0 & 0 & \sqrt{(1-\beta_0)(\tau_2-\tau_1)}\boldsymbol{R}_2\boldsymbol{A}_{\mathrm{d}} & 0 \end{bmatrix}$$



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$$\Pi_{41}^{(1)} = \begin{bmatrix} \sqrt{\tau_1}\tilde{N}^{\mathrm{T}} \\ \sqrt{(\tau_2-\tau_1)}\tilde{V}^{\mathrm{T}} \end{bmatrix}, \quad \Pi_{41}^{(2)} = \begin{bmatrix} \sqrt{\tau_1}\tilde{N}^{\mathrm{T}} \\ \sqrt{(\tau_2-\tau_1)}\tilde{S}^{\mathrm{T}} \end{bmatrix}$$

$$\Pi_{41}^{(3)} = \begin{bmatrix} \sqrt{\tau_1}\tilde{M}^{\mathrm{T}} \\ \sqrt{(\tau_2-\tau_1)}\tilde{V}^{\mathrm{T}} \end{bmatrix}, \quad \Pi_{41}^{(2)} = \begin{bmatrix} \sqrt{\tau_1}\tilde{M}^{\mathrm{T}} \\ \sqrt{(\tau_2-\tau_1)}\tilde{S}^{\mathrm{T}} \end{bmatrix}$$

$$\Lambda_1 = \boldsymbol{P}_1\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \boldsymbol{N}_{10} + \boldsymbol{N}_{10}^{\mathrm{T}} + \bar{B}_{\mathrm{f}}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}$$
$$\Lambda_2 = \bar{P}_3\boldsymbol{A} + \bar{A}_{\mathrm{f}}^{\mathrm{T}} + \boldsymbol{N}_{11} + \bar{B}_{\mathrm{f}}\boldsymbol{C}$$
$$\Lambda_3 = \beta_0(\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}) + \boldsymbol{N}_2 - \boldsymbol{N}_{10}^{\mathrm{T}} + \boldsymbol{M}_{10}^{\mathrm{T}}$$
$$\Lambda_4 = \beta_0(\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\bar{P}_3 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}) - \boldsymbol{N}_{11}^{\mathrm{T}} + \boldsymbol{M}_{11}^{\mathrm{T}}$$
$$\Lambda_5 = -\boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}} + \boldsymbol{M}_2 + \boldsymbol{M}_2^{\mathrm{T}}$$
$$\Lambda_6 = -\boldsymbol{N}_3 + \boldsymbol{M}_3 - \boldsymbol{M}_2^{\mathrm{T}} + \boldsymbol{V}_2^{\mathrm{T}}$$
$$\Lambda_7 = -\boldsymbol{Q}_1 - \boldsymbol{M}_3 - \boldsymbol{M}_3^{\mathrm{T}} + \boldsymbol{V}_3 + \boldsymbol{V}_3^{\mathrm{T}}$$
$$\Lambda_8 = (1-\beta_0)(\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}) + \boldsymbol{N}_4 - \boldsymbol{V}_{10}^{\mathrm{T}} + \boldsymbol{S}_{10}^{\mathrm{T}}$$
$$\Lambda_9 = (1-\beta_0)(\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\bar{P}_3 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}) - \boldsymbol{V}_{11}^{\mathrm{T}} + \boldsymbol{S}_{11}^{\mathrm{T}}$$
$$\Lambda_{10} = -\boldsymbol{N}_4 + \boldsymbol{M}_4 - \boldsymbol{V}_2^{\mathrm{T}} + \boldsymbol{S}_2^{\mathrm{T}}$$
$$\Lambda_{11} = -\boldsymbol{M}_4 + \boldsymbol{V}_4 - \boldsymbol{V}_3^{\mathrm{T}} + \boldsymbol{S}_3^{\mathrm{T}}$$
$$\Lambda_{12} = -\boldsymbol{V}_4 - \boldsymbol{V}_4^{\mathrm{T}} + \boldsymbol{S}_4 + \boldsymbol{S}_4^{\mathrm{T}}$$
$$\tilde{N}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{N}_{10}^{\mathrm{T}} & \boldsymbol{N}_{11}^{\mathrm{T}} & \boldsymbol{N}_2^{\mathrm{T}} & \boldsymbol{N}_3^{\mathrm{T}} & \boldsymbol{N}_4^{\mathrm{T}} & \boldsymbol{N}_5^{\mathrm{T}} \end{bmatrix}$$
$$\tilde{M}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{M}_{10}^{\mathrm{T}} & \boldsymbol{M}_{11}^{\mathrm{T}} & \boldsymbol{M}_2^{\mathrm{T}} & \boldsymbol{M}_3^{\mathrm{T}} & \boldsymbol{M}_4^{\mathrm{T}} & \boldsymbol{M}_5^{\mathrm{T}} \end{bmatrix}$$

and $\boldsymbol{\Xi}_{33}$, $\boldsymbol{\Xi}_{44}$ are defined as the same in Theorem 1.

Moreover, a suitable filter of the form (16) and(17) is given as

$$\begin{cases} \boldsymbol{A}_{\mathrm{f}} = \bar{A}_{\mathrm{f}}\bar{P}_3^{-1} \\ \boldsymbol{B}_{\mathrm{f}} = \bar{B}_{\mathrm{f}} \\ \boldsymbol{C}_{\mathrm{f}} = \bar{C}_{\mathrm{f}}\bar{P}_3^{-1} \\ \boldsymbol{D}_{\mathrm{f}} = \bar{D}_{\mathrm{f}} \end{cases} \tag{57}$$

*Proof:* Since $\bar{P}_3 > 0$, there exist non-singular matrix $\boldsymbol{P}_2$ and $\boldsymbol{P}_3 > 0$ such that $\bar{P}_3 = \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{P}_3^{-1}\boldsymbol{P}_2$. Defining

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2^{\mathrm{T}} \\ \boldsymbol{P}_2 & \boldsymbol{P}_3 \end{bmatrix}, \quad J = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{P}_3^{-1} \end{bmatrix} \tag{58}$$

it is easy to see that $\boldsymbol{P} > 0$ is equivalent to $\boldsymbol{P}_1 - \bar{P}_3 = \boldsymbol{P}_1 - \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{P}_3^{-1}\boldsymbol{P}_2 > 0$.

Pre- and post-multiplying (27) with $\Pi = \mathrm{diag}\{J, \underbrace{\boldsymbol{I}, \boldsymbol{I}, \ldots, \boldsymbol{I}}_{13}\}$ and its transpose and letting

$$\begin{cases} \bar{A}_{\mathrm{f}} = \hat{A}_{\mathrm{f}}\bar{P}_3, \quad \hat{A}_{\mathrm{f}} = \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{A}_{\mathrm{f}}\boldsymbol{P}_2^{-\mathrm{T}} \\ \bar{B}_{\mathrm{f}} = \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{B}_{\mathrm{f}} \\ \bar{C}_{\mathrm{f}} = \hat{C}_{\mathrm{f}}\bar{P}_3, \quad \hat{C}_{\mathrm{f}} = \boldsymbol{C}_{\mathrm{f}}\boldsymbol{P}_2^{-\mathrm{T}} \\ \bar{D}_{\mathrm{f}} = \boldsymbol{D}_{\mathrm{f}} \\ \boldsymbol{N}_1^{\mathrm{T}}J^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{N}_{10}^{\mathrm{T}} & \boldsymbol{N}_{11}^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{M}_1^{\mathrm{T}}J^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{M}_{10}^{\mathrm{T}} & \boldsymbol{M}_{11}^{\mathrm{T}} \end{bmatrix} \\ \boldsymbol{V}_1^{\mathrm{T}}J^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{V}_{10}^{\mathrm{T}} & \boldsymbol{V}_{11}^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{S}_1^{\mathrm{T}}J^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{S}_{10}^{\mathrm{T}} & \boldsymbol{S}_{11}^{\mathrm{T}} \end{bmatrix} \end{cases} \tag{59}$$

we can conclude (55).

Next, we will show that, if (55) and (56) are solvable for $\bar{A}_{\mathrm{f}}$, $\bar{B}_{\mathrm{f}}$, $\bar{C}_{\mathrm{f}}$, $\bar{D}_{\mathrm{f}}$ and $\bar{P}_3$, then, the parameter matrices of the filter (16) and(17) can be chosen as in (57).

Replacing $(\boldsymbol{A}_{\mathrm{f}}, \boldsymbol{B}_{\mathrm{f}}, \boldsymbol{C}_{\mathrm{f}}, \boldsymbol{D}_{\mathrm{f}})$ by $(\boldsymbol{P}_2^{-\mathrm{T}}\hat{A}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}, \boldsymbol{P}_2^{-\mathrm{T}}\bar{B}_{\mathrm{f}}, \hat{C}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}, \bar{D}_{\mathrm{f}})$ in (55) and then pre- and post-multiplying them with $\Pi$ and its transpose, we can also obtain (55), obviously $(\boldsymbol{P}_2^{-\mathrm{T}}\hat{A}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}, \boldsymbol{P}_2^{-\mathrm{T}}\bar{B}_{\mathrm{f}}, \hat{C}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}, \bar{D}_{\mathrm{f}})$ can be chosen as the filter parameters, that is, the following filter

$$\dot{\bar{x}}_{\mathrm{f}}(t) = \boldsymbol{P}_2^{-\mathrm{T}}\hat{A}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}\bar{x}_{\mathrm{f}}(t) + \boldsymbol{P}_2^{-\mathrm{T}}\bar{B}_{\mathrm{f}}\hat{y}(t) \tag{60}$$

$$\bar{z}_{\mathrm{f}}(t) = \hat{C}_{\mathrm{f}}\boldsymbol{P}_2^{\mathrm{T}}\bar{x}_{\mathrm{f}}(t) + \bar{D}_{\mathrm{f}}\hat{y}(t) \tag{61}$$

can guarantee the filtering-error system (18) and (19) is asymptotically stable with the $H_\infty$ performance bound $\gamma$. Defining $x_{\mathrm{f}}(t) = \boldsymbol{P}_2^{\mathrm{T}}\bar{x}_{\mathrm{f}}(t)$, (60) and (61) become

$$\dot{x}_{\mathrm{f}}(t) = \hat{A}_{\mathrm{f}}x_{\mathrm{f}}(t) + \bar{B}_{\mathrm{f}}\hat{y}(t) \tag{62}$$

$$z_{\mathrm{f}}(t) = \hat{C}_{\mathrm{f}}x_{\mathrm{f}}(t) + \bar{D}_{\mathrm{f}}\hat{y}(t) \tag{63}$$

Then, we can complete the proof. □

*Remark 4:* From Theorems 1 and 2, it can be seen that the feasibility of LMIs (55) and (56) depend on not only $\tau_1$ and $\tau_2$ but also the probability distribution of the delay taking values in the interval. Therefore more information of the time delay is involved in (55) and (56) that may lead to a larger allowable upper bound of the time delay.

Similarly, the following result can be obtained for the stabilisation of the systems (47) and (48).

*Corollary 2:* The systems (47) and (48) are asymptotically stable if there exist matrices $\boldsymbol{P}_1 > 0$, $\bar{P}_3 > 0$, $\boldsymbol{Q}_2 > 0$, $\boldsymbol{R}_2 > 0$, $\boldsymbol{N}_{10}$, $\boldsymbol{N}_{11}$, $\boldsymbol{S}_{10}$, $\boldsymbol{S}_{11}$, $\boldsymbol{N}_2$, $\boldsymbol{S}_2$ and matrices $\bar{A}_{\mathrm{f}}$, $\bar{B}_{\mathrm{f}}$, $\bar{C}_{\mathrm{f}}$, $\boldsymbol{D}_{\mathrm{f}}$ of appropriate dimensions such that the following LMIs hold

$$\hat{\Psi} = \begin{bmatrix} \hat{\Psi}_{11} & * & * \\ \hat{\Psi}_{21} & \Psi_{22} & * \\ \hat{\Psi}_{31}^{(l)} & \Psi_{32}^{(l)} & \Psi_{33} \end{bmatrix} < 0, \quad l = 1, 2 \tag{64}$$

$$\boldsymbol{P}_1 - \bar{P}_3 > 0 \tag{65}$$

**Table 1** Allowable upper bound of $\tau_2$ with $\gamma = 0.43$

| $\tau_1$ | 0.05 | 0.20 | 0.40 | 0.60 | 0.80 | 1 |
|---|---|---|---|---|---|---|
| $\beta_0 = 0.1$ | 1.06 | 1.06 | 1.06 | 1.05 | 1.04 | 1.03 |
| $\beta_0 = 0.5$ | 1.52 | 1.51 | 1.47 | 1.38 | 1.23 | 1.04 |
| $\beta_0 = 0.9$ | 5.07 | 4.91 | 4.52 | 3.81 | 2.73 | 1.06 |

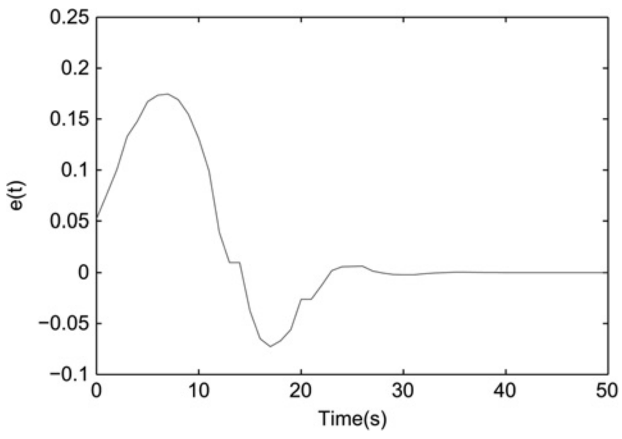


**Fig. 1** *Estimated signals error $e(t) = z(t) - z_f(t)$*

where (see equations at the bottom of the page)

and $\Psi_{22}$, $\Psi_{32}^{(l)}$, $\Psi_{33}$ are defined as the same in Corollary 1. Moreover, a suitable filter of the form (16) and (17) is given as (57).

## 4 Numerical example

*Example 1:* Consider the systems (2)–(4) with parameter matrices [23]

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \; \boldsymbol{A}_{\mathrm{d}} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} K, \; \boldsymbol{B} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \; \boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\boldsymbol{C}_{\mathrm{d}} = 0.2K, \; \boldsymbol{D} = 1, \; \boldsymbol{L} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; \boldsymbol{L}_{\mathrm{d}} = 0.1K, \; \boldsymbol{L}_{w} = 0.2$$
$$K = \begin{bmatrix} -3.75 & -11.5 \end{bmatrix} \tag{66}$$

To the same $\gamma = 0.43$ in Jiang and Han [23], the upper-delay bound is $\tau_2 = 0.9412$ when the lower-delay bound is $\tau_0 = 0.2$. By using Theorem 2 with different $\beta_0$ and $\tau_1$, the computation results for the allowable upper bound $\tau_2$ are given in Table 1.

For example when $\gamma = 0.43$, $\beta_0 = 0.5$, $\tau_1 = 0.2$, $\tau_2 = 1.51$, we can obtain the parameter matrices of the filter

$$\boldsymbol{A}_{\mathrm{f}} = \begin{bmatrix} 0.0763 & 0.2459 \\ -1.7777 & -0.6578 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{f}} = \begin{bmatrix} 0.0197 \\ -0.1243 \end{bmatrix}$$

$$\boldsymbol{C}_{\mathrm{f}} = \begin{bmatrix} -0.9957 & -0.7552 \end{bmatrix}, \quad \boldsymbol{D}_{\mathrm{f}} = 0.4992$$

To illustrate the performance of the designed filter, choose the disturbance function as follows

$$\omega(t) = \begin{cases} -0.1, & 5s \le t \le 10s \\ 0.2, & 15s \le t \le 20s \\ 0, & \text{otherwise} \end{cases}$$

Fig. 1 shows the error-estimation signal of $e(t) = z(t) - z_{\mathrm{f}}(t)$, Fig. 2 shows the state of filtering-error system with the initial values $x(0) = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$.

This example was also considered by [24], when the upper-delay bound $\tau_2 = 2.20$, the lower-delay bound $\tau_0 = 0.2$ and $\gamma = 0.43$. Whereas our lower-delay bound $\tau_0 = 0$, obviously, our result is much better than that obtained in [23, 24].

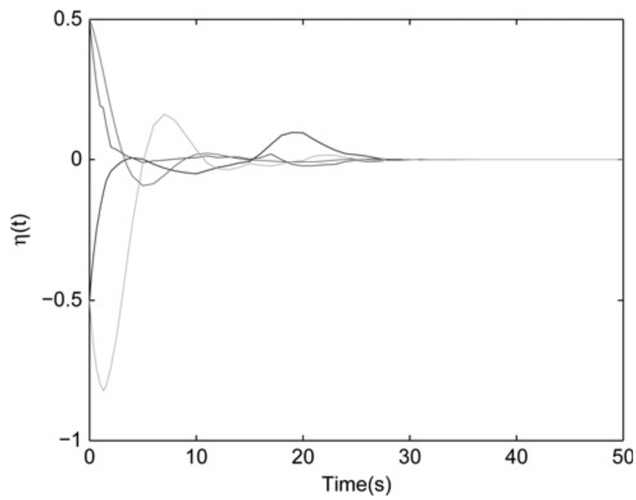


**Fig. 2** *State curves of filtering-error system*

*Example 2:* Consider the system (47) and (48) with parameter matrices as (66).

When $\gamma = 0.43$, by using Corollary 2, we can obtain the upper bound of time delay $\tau_2 = 1.0032$, and the parameter matrices of the filter are obtained as follows

$$\boldsymbol{A}_{\mathrm{f}} = \begin{bmatrix} 0.1329 & 0.3800 \\ -2.0929 & -1.3817 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{f}} = \begin{bmatrix} 0.0378 \\ -0.1506 \end{bmatrix}$$

$$\boldsymbol{C}_{\mathrm{f}} = \begin{bmatrix} -1.3433 & -1.5543 \end{bmatrix}, \quad \boldsymbol{D}_{\mathrm{f}} = 0.4359$$

*Example 3:* Now, we consider a mechanical system borrowed from [32] with small modifications, shown in Fig. 3.

In this system, $x_1$ and $x_2$ are the positions of masses $m_1$ and $m_2$, respectively, and $k_1$ and $k_2$ are the spring constants. The viscous friction coefficient between the massed and the horizontal surface is denoted by $c$. A state-space realisation

$$\hat{\Psi}_{11} = \begin{bmatrix} \Theta_1 & * \\ \bar{P}_3\boldsymbol{A} + \bar{A}_{\mathrm{f}}^{\mathrm{T}} + \boldsymbol{N}_{11} + \bar{B}_{\mathrm{f}}\boldsymbol{C} & \bar{A}_{\mathrm{f}} + \bar{A}_{\mathrm{f}}^{\mathrm{T}} \end{bmatrix}$$

$$\hat{\Psi}_{21} = \begin{bmatrix} \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} + \boldsymbol{N}_2 - \boldsymbol{N}_{10}^{\mathrm{T}} + \boldsymbol{S}_{10}^{\mathrm{T}} & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\bar{P}_3 + \boldsymbol{C}_{\mathrm{d}}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} - \boldsymbol{N}_{11}^{\mathrm{T}} + \boldsymbol{S}_{11}^{\mathrm{T}} \\ \boldsymbol{N}_3 - \boldsymbol{S}_{10}^{\mathrm{T}} & -\boldsymbol{S}_{11}^{\mathrm{T}} \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{D}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} & \boldsymbol{B}^{\mathrm{T}}\bar{P}_3 + \boldsymbol{D}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}} \\ \sqrt{\tau_2}\boldsymbol{R}_2\boldsymbol{A} & 0 \\ \boldsymbol{L} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{C} & -\bar{C}_{\mathrm{f}} \end{bmatrix}$$

$$\hat{\Psi}_{31}^{(1)} = \begin{bmatrix} \sqrt{\tau_2}\boldsymbol{N}_{10}^{\mathrm{T}} & \sqrt{\tau_2}\boldsymbol{N}_{11}^{\mathrm{T}} \end{bmatrix}, \quad \hat{\Psi}_{31}^{(2)} = \begin{bmatrix} \sqrt{\tau_2}\boldsymbol{S}_{10}^{\mathrm{T}} & \sqrt{\tau_2}\boldsymbol{S}_{11}^{\mathrm{T}} \end{bmatrix}$$

$$\Theta_1 = \boldsymbol{P}_1\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{Q}_2 + \boldsymbol{N}_{10} + \boldsymbol{N}_{10}^{\mathrm{T}} + \bar{B}_{\mathrm{f}}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\bar{B}_{\mathrm{f}}^{\mathrm{T}}$$




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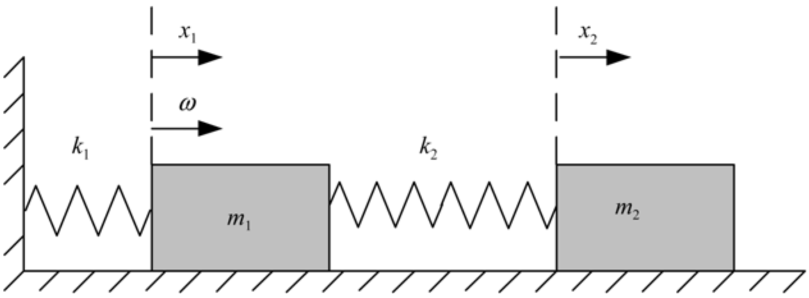


**Fig. 3** *Mass-spring system*

of this system is given by the equations in (2)–(4) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{c}{m_1} & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & -\dfrac{c}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$
$$D = d, \quad A_d = C_d = L_d = L_\omega = 0$$

where $d$ is a constant.

It is assumed that the position of mass $m_1$ is measured by a device with disturbance $\omega(t)$. The parameters are chosen as the same in [32] of $m_1 = 1$, $m_2 = 0.5$, $k_1 = k_2 = 1$, $c = 0.5$ and $d = 0.1$. Thus, the matrices for the equations of (2)–(4) are given as follows

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.5 & 0 \\ 2 & -2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

By using Theorem 2, we can obtain the following filtering parameters of (57) when $\gamma = 1$

$$A_f = \begin{bmatrix} -0.7442 & 2.3550 & 23.2524 & -0.1325 \\ -0.0433 & -0.9939 & -0.9366 & 2.5538 \\ -0.9831 & -0.0498 & -0.6931 & 0.0252 \\ 0.0540 & -0.7471 & 0.0728 & -0.4414 \end{bmatrix}$$

$$B_f = \begin{bmatrix} -0.5187 \\ -0.8864 \\ -11.0284 \\ -1.2007 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -0.0117 & -0.4837 & -0.0650 & 0.2586 \end{bmatrix}$$
$$D_f = 0.0769$$

## 5 Conclusion

This paper has investigated the problem of filtering design for linear time-delay system. Different from the common assumption in the existing references, the time delay considered here is fast time-varying delay satisfying a certain stochastic characteristic, and the probability distribution of the delay taking values in some intervals is assumed to be known a priori. Corresponding to the probability of the delay taking value in different intervals, a stochastic variable satisfying Bernoulli random binary distribution has been introduced and a new system model was built by employing the information of the probability distribution. Then delay-distribution-dependent criteria has been derived via filtering-error system. It should be noted that the solvability of the obtained criteria depend not only on the size of the delay, but also the probability distribution of it. Examples have been given to illustrate the feasibility and effectiveness of the proposed method.

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